ATTRACTIVITY RESULTS AND EXTREMAL SOLUTIONS FOR FRACTIONAL ORDER NONLINEAR INTEGRO-DIFFERENTIAL EQUATION

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ABSTRACT

In this paper, we study the existence the solution as well as locally attractivity results for fractional order nonlinear functional integro- differential equation in Banach algebras under mixed lipschitz and caratheodory conditions by using hybrid fixed point theorem. The existence of extremal solutions is also proved under certain monotonicty conditions. The results are illustrated by a concrete example.

KEYWORDS:

Functional integro-differential equation, Banach Algebras, Fixed point theorem, Existence and Extremal solutions, locally attractivitysolutions.

Introduction: Fractional calculus is generalization of the deifferetniationsanfd integrations of the arbitrary non-integers order. The concept of the fractional calculusare arising in the mathematical modeling of system and process occurring in many enginnering and scirntificapprocahses such as physics ,chemistry aerodynamics electrodynamics,econonmics, etc[11,12,13].Nonlinear fractional integro-differetnial equations are an important class to solve of the various type of the equations have been stuidedby many researcher using the different technique.[14].Dhage are introduced and proved algorthims for the existence solution for nonlinear first order ordinary integro-differetnial equations and approxiations solutions for intial value problems.[15]

In this paper we study the existence result is obtained for fractional order nonlinear integro-differential Equation by using a hybrid fixed point theorem of three operators in Banach algebras due to B.C.Dhage[4]

Statement of the problem:

Let $\mathbb{R}$ denote the real line and $\mathbb{R}_+$ be the set of nonnegative real numbers, that is,$\mathbb{R}_+ = [0, \infty)$. Let $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$ denote the class of real-valued functions defined and continuous on $\mathbb{R}_+$.Consider the fractional order nonlinear functional integro- differential equation(in short FNFIDE)

$$\left. \begin{array}{c} \dfrac{d^\alpha}{dt^\alpha}\left[\dfrac{x(t)-\sum_{i=1}^{i=n} I^{\beta_i} q_i\big(t,x(t)\big)}{f\big(t,x(t),x(u(t))\big)}\right] = g\left(t,x(t),\int_0^t k\big(t,x(s),x(v(s))\big)\,ds\right) \\ x(0) = 0 \end{array} \right\} (1.1)$$

Where $D^\alpha = \frac{d^\alpha}{dt^\alpha}$ denote Riemann – Liouville derivative of order α, $1 < \alpha < 2$ and I^{β_i} is the Riemann – Liouville integral of order $\beta_i, i = 1,2,3, \dots n$; $u, v: \mathbb{R}_+ \to \mathbb{R}, f: \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$, $g: \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $k: \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $q_i: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ are continuous functions with $q_i(0,0) = 0, i = 1,2,3, \dots n$.

By a solution of the FNFIDE (1.1) we mean a function $x \in \mathcal{BC}(\mathbb{R}_+, \mathbb{R})$ if

(i) The function $t \to \frac{x(t) - \sum_{i=1}^{i=n} I^{\beta_i} q_{i(t,x(t))}}{f(t,x(t),x(u(t)))}$ is Riemann – Liouville differentiable, and

(ii) x satisfies the relations in (1.1) on $\mathbb{R}_+$.

where $\mathcal{BC}(\mathbb{R}_+, \mathbb{R})$ is the space of continuous and bounded real-valued functions defined on $\mathbb{R}_+$.

In this paper, we prove the existence the solution for FNFIDE (1.1) employing a classical hybrid fixed point theorem of Dhage [4]. In the next section, we collect some preliminary definitions and results.

2 Preliminaries:

In this section we give the definitions, notation, hypothesis and preliminary tools, which will be used in the sequel.

Let $\mathbb{X} = \mathcal{BC}(\mathbb{R}_+, \mathbb{R})$ be Banach algebra with norm $\|.\|$ and Ω be a subset of $\mathbb{X}$. Let a mapping $\mathbb{A}: \mathbb{X} \to \mathbb{X}$ be an operator and consider the following operator equation in $\mathbb{X}$, namely,

$x(t) = (\mathbb{A}x)(t)$, for all $t \in \mathbb{R}_+$ (2.1)

Below we give some different characterization of the solutions for operator equation (2.1) on $\mathbb{R}_+$. We need the following definitions.s

Definition 2.1 [8]: We say that solution of the equation (2.1) are locally attractive if there exists a closed ball $\overline{B_r(0)}$ in the space $\mathcal{BC}(\mathbb{R}_+, \mathbb{R})$ and for some real number $r > 0$ such that for arbitrary solution $x = x(t)$ and $y = y(t)$ of equation (2.1) belonging to $\overline{B_r(0)} \cap \Omega$ we have that

$\lim_{t \to \infty} (x(t) - y(t)) = 0$ (2.2)

Definition 2.2[5]: Let $\mathbb{X}$ be a Banach space. A mapping $\mathbb{A}: \mathbb{X} \to \mathbb{X}$ is called Lipschitz if there is a constant $\alpha > 0$ such that, $\|\mathbb{A}x - \mathbb{A}y\| \le \alpha \|x - y\|$ for all $x, y \in \mathbb{X}$. If $\alpha < 1$, then $\mathbb{A}$ is called a contraction on $\mathbb{X}$ with the contraction constant α.

Definition 2.3 [6](Dugundji and Granas (1982)): An operator $\mathbb{Q}$ on a Banach space $\mathbb{X}$ into itself is called compact if for any bounded subset $\mathbb{S}$ of $\mathbb{X}$, $\mathbb{Q}(\mathbb{S})$ is relatively compact subset of $\mathbb{X}$. If $\mathbb{Q}$ is continuous and compact, then it is called completely continuous on $\mathbb{X}$.

Definition 2.4[5]: Let $\mathbb{X}$ be a Banach space with the norm $\|\cdot\|$ and let $\mathbb{Q}: \mathbb{X} \to \mathbb{X}$, be an operator (in general nonlinear). Then $\mathbb{Q}$ is called

   i.   Compact if $\mathbb{Q}(\mathbb{X})$ is relatively compact subset of $\mathbb{X}$.

   ii.   Totally compact if $\mathbb{Q}(\mathbb{S})$ is totally bounded subset of $\mathbb{X}$ for any bounded subset $\mathbb{S}$ of $\mathbb{X}$.

   iii.   Completely continuous if it is continuous and totally bounded operator on $\mathbb{X}$.

It is clear that every compact operator is totally bounded but the converse need not be true. We recall the basic definitions of fractional calculus which are useful in what follows.

Definition 2.5 [9]: The Riemann – Liouville fractional derivative of order $\xi > 0, n - 1 < \xi < n, n \in \mathcal{N}$ with lower limit zero for a function f is defined as $\mathfrak{D}^\xi f(t) = \frac{1}{\Gamma(\xi)} \frac{d}{dt} \int_0^t \frac{f(s)}{(t-s)^\xi} ds$, $t > 0$ Such that

$\mathfrak{D}^{-\xi} f(t) = I^{\xi} f(t) = \frac{1}{\Gamma(\xi)} \int_0^t \frac{f(s)}{(t-s)^{1-\xi}} ds$ respectively.

Definition 2.6[9]: The Riemann-Liouville fractional integral of order $\xi > 0, n - 1 < \xi < n, n \in \mathcal{N}$ with lower limit zero for a function f is defined by the formula: $I^{\xi} f(t) = \frac{1}{\Gamma(\xi)} \int_0^t \frac{f(s)}{(t-s)^{1-\xi}} ds$, $t > 0$

where $\Gamma(\xi)$ denote the Euler gamma function. The Riemann-Liouville fractional derivative operator of order ξ defined by $\mathfrak{D}^{\xi} = \frac{d^{\xi}}{dt^{\xi}} = \frac{d}{dt} \circ I^{1-\xi}$.

Theorem 2.1[5]: (Arzela-Ascoli Theorem) If every uniformly bounded and equicontinuous sequence $\{f_n\}$ of functions in $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$, then it has a convergent subsequence.

Theorem 2.2[5]: A metric space X is compact iff every sequence in X has a convergent subsequence.

The Important combination of a metric and a topological fixed point theorem involving three operators in Banach algebras in its important from than that appeared in Dhage[1].We employ a new hybrid fixed pint theorem proved by Dhage [2] which is the main tool in the existence theorem of solutions of FNFIDE.

Theorem 2.3[4]: Let $\mathbb{S}$ be a non empty, convex, closed and bounded subset of the Banach space $\mathbb{X}$ and let $\mathbb{A}, \mathbb{C}: \mathbb{X} \to \mathbb{X}$ and $\mathbb{B}: \mathbb{S} \to \mathbb{X}$ are three operators satisfying:

a) $\mathbb{A}$ and $\mathbb{C}$ are Lipschitzian with lipschitz constants ζ, η respectively.

b) $\mathbb{B}$ is completely continuous, and

c) $x = \mathbb{A}x\mathbb{B}y + \mathbb{C}x \in \mathbb{S}$ for all $x, y \in \mathbb{S}$

d) $\zeta M + \eta < 1$ where $M = \|\mathbb{B}(\mathbb{S})\| = sup\{\|\mathbb{B}x\|: x \in \mathbb{S}\}$

Then the operator equation $x = \mathbb{A}x\mathbb{B}y + \mathbb{C}x$ has a solution in $\mathbb{S}$.

3.Existence Theory:

We seek the solution of (2.1) in the space $\mathcal{BC}(\mathbb{R}_+, \mathbb{R})$ of continuous and real – valued function defined on $\mathbb{R}_+$. Define a standard norm $\|\cdot\|$ and a multiplication " $\cdot$ " in $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$ by,

$$\|x\| = sup\{|x(t)|: t \in \mathbb{R}_+\}, \quad (xy)(t) = x(t)y(t), \quad t \in \mathbb{R}_+$$

Clearly, $\mathcal{BC}(\mathbb{R}_+, \mathbb{R})$ becomes a Banach space with respect to the above norm and the multiplication in it.

Definition 3.1[9]: A mapping $g: \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is Caratheodory if:

i) $t \to g(t, x, y)$ is measurable for each $x, y \in \mathbb{R}$ and

ii) $(x, y) \to g(t, x, y)$ is continuous almost everywhere for $t \in \mathbb{R}_+$.

Furthermore a Caratheodory function g is $\mathcal{L}^1$ −Caratheodory if:

iii) For each real number $r > 0$ there exists a function $h_r \in \mathcal{L}^1(\mathbb{R}_+, \mathbb{R})$ such that $|g(t, x, y)| \leq h_r(t) a.e.\ t \in \mathbb{R}_+$ for all $x, y \in \mathbb{R}$ with $|x|_r \leq r$ and $|y|_r \leq r$.

Finally a caratheodory function g is $\mathcal{L}^1_{\mathbb{X}}$ −caratheodory if:

iv) There exists a function $h \in \mathcal{L}^1(\mathbb{R}_+, \mathbb{R})$ such that $|g(t, x, y)| \leq h(t), \quad a.e.\ t \in \mathbb{R}_+$ for all $x, y \in \mathbb{R}$.

For convenience, the function $\hbar$ is referred to as a bound function for g.

Lemma 3.1: Suppose that $\alpha, \beta \in (1,2)$ and the function $f, g, q_i, i = 1,2,3, \ldots n$ satisfying FNFIDE(1.1).

Then a function $x \in \mathcal{BC}(\mathbb{R}_+, \mathbb{R})$ is a solution of the FNFIDE(1.1) if and only if it a solution of the nonlinear integral equation

$$x(t) = f\left(t, x(t), x(u(t))\right)\left[\frac{1}{\Gamma(\alpha)}\int_0^t \frac{g\left(s, x(s), \int_0^t k\left(s, x(\tau), x(v(\tau))\right)d\tau\right)ds}{(t-s)^{1-\alpha}}\right]$$

$+\sum_{i=1}^{i=n} I^{\beta_i} q_i\left(t, x(t)\right), t \in \mathbb{R}_+ \qquad (3.1)$

Proof: Integrating equation (1.1) of fractional order α w.r.t. t, we get,

$$I^\alpha D^\alpha\left[\frac{x(t) - \sum_{i=1}^{i=n} I^{\beta_i} q_i\left(t, x(t)\right)}{f\left(t, x(t), x(u(t))\right)}\right] = I^\alpha\left[g\left(t, x(t), \int_0^t k\left(t, x(s), x(v(s))\right)ds\right)\right]$$

$$\left[\frac{x(t) - \sum_{i=1}^{i=n} I^{\beta_i} q_i\left(t, x(t)\right)}{f\left(t, x(t), x(u(t))\right)}\right]_0^t = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{g\left(s, x(s), \int_0^t k\left(s, x(\tau), x(v(\tau))\right)d\tau\right)ds}{(t-s)^{1-\alpha}}$$

$$\frac{x(t) - \sum_{i=1}^{i=n} I^{\beta_i} q_i(t, x(t))}{f\left(t, x(t), x(u(t))\right)} - \frac{x(0) - \sum_{i=1}^{i=n} I^{\beta_i} q_i(0, x(0))}{f\left(t, x(t), x(u(0))\right)} = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{g\left(s, x(s), \int_0^t k\left(s, x(\tau), x(v(\tau))\right)d\tau\right)ds}{(t-s)^{1-\alpha}}$$

Since $x(0) = 0, q_i(0,0) = 0, f(0,0,0) \neq 0$

It follows that

$$x(t) = f\left(t, x(t), x(u(t))\right)\left[\frac{1}{\Gamma(\alpha)}\int_0^t \frac{g\left(s, x(s), \int_0^t k\left(s, x(\tau), x(v(\tau))\right)d\tau\right)ds}{(t-s)^{1-\alpha}}\right]$$

$$+\sum_{i=1}^{i=n} I^{\beta_i} q_i\left(t, x(t)\right), t \in \mathbb{R}_+$$

Conversely differentiate (3.1) of order α with respect to t, we get

$$D^\alpha\left[\frac{x(t) - \sum_{i=1}^{i=n} I^{\beta_i} q_i(t, x(t))}{f\left(t, x(t), x(u(t))\right)}\right] = D^\alpha\left[\frac{1}{\Gamma(\alpha)}\int_0^t \frac{g\left(t, x(s), \int_0^t k\left(s, x(\tau), x(v(\tau))\right)d\tau\right)ds}{(t-s)^{1-\alpha}}\right]$$

$$D^\alpha\left[\frac{x(t) - \sum_{i=1}^{i=n} I^{\beta_i} q_i(t, x(t))}{f\left(t, x(t), x(u(t))\right)}\right] = D^\alpha I^\alpha\left[g\left(t, x(t), \int_0^t k\left(t, x(s), x(v(s))\right)ds\right)\right]$$

$$D^\alpha\left[\frac{x(t) - \sum_{i=1}^{i=n} I^{\beta_i} q_i(t, x(t))}{f\left(t, x(t), x(u(t))\right)}\right] = \left[g\left(t, x(t), \int_0^t k\left(t, x(s), x(v(s))\right)ds\right)\right]$$

We consider the fractional nonlinear integro-differential equation(1.1) assuming that the following hypotheses are satisfied.

$(\mathcal{H}_1)$ The function $u: \mathbb{R}_+ \to \mathbb{R}_+$ are continuous function with $u(0) = 0$ and the function $v: \mathbb{R}_+ \to \mathbb{R}_+$
is measurable.

$(\mathcal{H}_2)$ The function $q_i: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}, i = 1,2,3 \ldots \ldots n$, with $q_i(0,0,) = 0, i = 1,2,3 \ldots \ldots n$ are continuous and there exist positive functions $\lambda_i, i = 1,2,3 \ldots \ldots n$ with bound $\|\lambda_i\|$ such that

$$\left|q_i\left(t, x(t)\right) - q_i\left(t, y(t)\right)\right| \leq \lambda_i(t)|x(t) - y(t)|, \forall t \in \mathbb{R}_+, x, y \in \mathbb{R}$$

$\left|q_i\left(t, x(t)\right)\right| \leq Q_i(t)$ and $\lim_{t\to\infty} Q_i(t) = 0$

$(\mathcal{H}_3)$ The function $f: \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}\backslash\{0\}$ is continuous and bounded with bound $\mathbb{F} = sup_{(t, x_1, x_2)}|f(t, x_1, x_2)|$ there exist a bounded function $\ell: \mathbb{R}_+ \to \mathbb{R}$ with bound $\mathbb{L}$ such that

$$\left| \begin{matrix} f\left(t, x_1(t), x_2\big(u(t)\big)\right) \\ -f\left(t, y_1(t), y_2\big(u(t)\big)\right) \end{matrix} \right| \leq \ell(t) max\{|x_1 - y_1|, |x_2 - y_2|\}$$

for all $t \in \mathbb{R}_+$ and $x, y \in \mathbb{R}$

($\mathcal{H}_4$) The function $k: \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$is continuous and there exist a function $\mu \in L^1(\mathbb{R}_+, \mathbb{R}_+)$ such that $\left| k\left(t, x(s), x\big(v(s)\big)\right) \right| \leq \mu(s)|x|$ for all $t, s \in \mathbb{R}_+$and $\in \mathbb{R}$.

($\mathcal{H}_5$)The function $g: \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfy Caratheodory conditions and there exist a function$h(t): \mathbb{R}_+ \to \mathbb{R}_+$ such that,

$$g\left(t, x(t), \int_0^t k\left(t, x(s), x\big(v(s)\big)\right) ds\right) \leq h(t) \ \forall t \in \mathbb{R}_+, x \in \mathbb{R}, \qquad (3.2)$$

($\mathcal{H}_6$)The function $v: \mathbb{R}_+ \to \mathbb{R}_+$ defined by the function $v(t) = \int_0^t \frac{h(t)}{(t-s)^{1-\alpha}} ds$ is bounded on $\mathbb{R}_+$.

Remark: Note that if the hypothesis ($\mathcal{H}_5$)hold, then there exist constant $K_1 > 0$ such that $\mathcal{K}_1 = \underbrace{sup}_{t \geq 0} \frac{1}{\Gamma(\alpha)} \int_0^t \frac{h(t)}{(t-s)^{1-\alpha}} ds$

1. Main Result:

Theorem 4.2: Assume that hypothesis ($\mathcal{H}_1 - \mathcal{H}_6$) holds then FNFIDE(1.1). Further if $\mathbb{L}\mathcal{K}_1 + \|\gamma\| < 1$ then FNFIDE (1.1)has a solution in the space $\mathcal{BC}(\mathbb{R}_+, \mathbb{R})$.Moreover, solution of the equation(1.1)S

Proof: Set $\mathbb{X} = \mathcal{BC}(\mathbb{R}_+, \mathbb{R})$ and define a subset $\mathbb{S}$ of $\mathbb{X}$ as$\mathbb{S} = \{x \in \mathbb{X}: \|x\| \leq r\}$.Where r satisfiesth inequality,

$$\|\alpha\|\mathcal{K}_1 + \|\gamma\| \leq r \qquad (4.1)$$

Clearly $\mathbb{S}$be a non empty, convex, closed and bounded subset of the Banach space $\mathbb{X}$. By lemma (3.1), problem (1.1) is equivalent to (3.1).

Now we define three operators $\mathbb{A}: \mathbb{X} \to \mathbb{X}$ and $\mathbb{B}: \mathbb{S} \to \mathbb{X}$ and $\mathbb{C}: \mathbb{X} \to \mathbb{X}$ by

$$\mathbb{A}x(t) = f\left(t, x(t), x\big(u(t)\big)\right), t \in \mathbb{R}_+ \quad (4.2)$$

$$\mathbb{B}x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{g\left(t, x(s), \int_0^t k\left(s, x(\tau), x\big(v(\tau)\big)\right) d\tau\right) ds}{(t-s)^{1-\alpha}}, t \in \mathbb{R}_+ \quad (4.3)$$

$$\mathbb{C}x(t) = \sum_{i=1}^{i=n} I^{\beta_i} q_i\left(t, x(t)\right), t \in \mathbb{R}_+ \quad (4.4)$$

$$\text{i.e.}\mathbb{C}x(t) = \sum_{i=1}^{n} \int_0^t \frac{(t-s)^{\beta_i-1}}{\Gamma(\beta_i)} q_i\left(s, x(s)\right) ds, t \in \mathbb{R}_+ \quad (4.5)$$

Then the integral equation (3.1) is equivalent to the operator equation
$$x(t) = \mathbb{A}x(t)\mathbb{B}x(t) + \mathbb{C}x(t), \forall t \in \mathbb{R}_+ \quad (4.6)$$
We shall show that, the operators $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ satisfy all the conditions of theorem (2.3).
This will be achieved in the following series of steps.

Step I: First show that $\mathbb{A}$ and $\mathbb{C}$ are lipschitzian on $\mathbb{X}$.

Let $x, y \in \mathbb{X}$, then by ($\mathcal{H}_3$) for $t \in \mathbb{R}_+$ we have,

$$|\mathbb{A}x(t) - \mathbb{A}y(t)| = \left| \left(t, x_1(t), x_2\big(u(t)\big)\right) - f\left(t, y_1(t), y_2\big(u(t)\big)\right) \right|$$

$$\leq \ell(t) max\{|x_1 - y_1|, |x_2 - y_2|\} \leq \mathbb{L}|x(t) - y(t)|$$

Taking the supremumover , we obtain

$\leq \mathbb{L}\|x - y\|$for all $x, y \in \mathbb{R}$.

Therefore $\mathbb{A}$ is lipschitzian with lipschitz constant$\zeta = \mathbb{L}$.

Next, we show that $\mathbb{C}$ is lipschitzian on $\mathbb{X}$. let $x, y \in \mathbb{X}$ be arbitrary, and then by hypothesis $(\mathcal{H}_2)$, we have,

$$
\begin{aligned}
|\mathbb{C}x(t) - \mathbb{C}y(t)| &= \left| \sum_{i=1}^{n} I^{\beta_i} q_i(t, x(t)) - \sum_{i=1}^{n} I^{\beta_i} q_i(t, y(t)) \right| \\
&\leq \left| \sum_{i=1}^{i=n} I^{\beta_i} [q_i(t, x(t)) - q_i(t, y(t))] \right| \\
&\leq \sum_{i=1}^{i=n} I^{\beta_i} |q_i(t, x(t)) - q_i(t, y(t))| \\
&\leq \sum_{i=1}^{i=n} I^{\beta_i} \lambda_i(t) |x(t) - y(t)| \\
&\leq \sum_{i=1}^{n} \int_0^t \frac{(t-s)^{\beta_i - 1}}{\Gamma(\beta_i)} \lambda_i(s) |x(s) - y(s)| \mathrm{ds} \\
&\leq \sum_{i=1}^{n} \int_0^t \frac{(t-s)^{\beta_i - 1}}{\Gamma(\beta_i)} \|\lambda_i\| |x(s) - y(s)| \mathrm{ds} \\
&\leq \|x - y\| \sum_{i=1}^{n} \int_0^t \frac{(t-s)^{\beta_i - 1}}{\Gamma(\beta_i)} \|\lambda_i\| \\
&\leq \|x - y\| \sum_{i=1}^{n} \left[-\frac{(t-s)^{\beta_i}}{\Gamma(\beta_i)\beta_i} \right]_0^t \|\lambda_i\| \\
&\leq \|x - y\| \sum_{i=1}^{n} \left[\frac{(t)^{\beta_i}}{\Gamma(\beta_i + 1)} \right] \|\lambda_i\| \\
&\leq \|x - y\| \sum_{i=1}^{n} \|\lambda_i\| \frac{\mathbb{T}^{\beta_i}}{\Gamma(\beta_i + 1)}
\end{aligned}
$$

This means that,

$$
\|\mathbb{C}x - \mathbb{C}y\| \leq \sum_{i=1}^{n} \|\lambda_i\| \frac{\mathbb{T}^{\beta_i}}{\Gamma(\beta_i + 1)} \|x - y\|
$$

$$
\|\mathbb{C}x - \mathbb{C}y\| \leq \|\gamma\| \|x - y\|
$$

Thus $\mathbb{C}$ is lipschitz on $\mathbb{X}$ with lipschitz constant constant $\eta = \|\gamma\| = \sum_{i=1}^{n} \|\lambda_i\| \frac{\mathbb{T}^{\beta_i}}{\Gamma(\beta_i + 1)}$.

Step II: To show the operator $\mathbb{B}$ is completely continuous on $\mathbb{X}$. Let $\{x_n\}$ be a sequence in $\mathbb{S}$ converging to a point x. Then by lebesgue dominated convergence theorem for all $t \in \mathbb{R}_+$, we obtain

$\lim_{n \to \infty} \mathbb{B}x_n(t)$

$$
= \lim_{n \to \infty} \left\{ \frac{1}{\Gamma(\alpha)} \int_0^t \frac{g\left(t, x_n(s), \int_0^t k\left(s, x_n(\tau), x_n(v(\tau))\right) d\tau\right) ds}{(t-s)^{1-\alpha}} ds \right\}
$$

$$= \left\{ \frac{1}{\Gamma(\alpha)} \int_0^t \frac{lim_{n\to\infty} \, g\left(t, x_n(s), \int_0^t k\left(s, x_n(\tau), x_n(v(\tau))\right) d\tau\right) ds}{(t-s)^{1-\alpha}} ds \right\}$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^t \frac{g\left(t, x(s), \int_0^t k\left(s, x(\tau), x(v(\tau))\right) d\tau\right) ds}{(t-s)^{1-\alpha}} ds$$

$$= \mathbb{B}x(t), \forall \, t \in \mathbb{R}_+$$

This shows that$\{\mathbb{B}x_n\}$ converges to $\mathbb{B}x$ pointwise on $\mathbb{S}$.

Next to show that sequence$\{\mathbb{B}x_n\}$ is an uniformly convergence in $\mathbb{S}$.

Let $t_1, t_2 \in \mathbb{R}_+$ be arbitrary with $t_1 < t_2$ then

$$|\mathbb{B}x_n(t_2) - \mathbb{B}x_n(t_1)|$$

$$= \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_2} \frac{g\left(t_2, x_n(s), \int_0^t k\left(s, x_n(\tau), x_n(v(\tau))\right) d\tau\right) ds}{(t_2 - s)^{1-\alpha}} ds \right.$$

$$\left. - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \frac{g\left(t_1, x_n(s), \int_0^t k\left(s, x_n(\tau), x_n(v(\tau))\right) d\tau\right) ds}{(t_1 - s)^{1-\alpha}} ds \right|$$

$$\leq \frac{1}{\Gamma(\alpha)} \left\{ \int_0^{t_2} \left| \frac{h(t_2)}{(t_2 - s)^{1-\alpha}} ds \right| - \int_0^{t_1} \left| \frac{h(t_1)}{(t_1 - s)^{1-\alpha}} ds \right| \right\}$$

$$\leq \frac{1}{\Gamma(\alpha)} \{v(t_2) - v(t_1)\}$$

$\to 0$ as $t_1 \to t_2, \forall n \in \mathcal{N}$

This shows that the convergence is uniform, by using property of uniform convergence that is uniform convergence imply continuity.

Hence $\mathbb{B}$ is continuous on $\mathbb{S}$.

Next we will prove that the set $\mathbb{B}(\mathbb{S})$ is uniformly bounded in $\mathbb{S}$, for any $x \in \mathbb{S}$, we have

$$|\mathbb{B}x(t)| = \left| \frac{1}{\Gamma(\alpha)} \int_0^t \frac{g\left(t, x(s), \int_0^t k\left(s, x(\tau), x(v(\tau))\right) d\tau\right) ds}{(t-s)^{1-\alpha}} ds \right|$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\left| g\left(t, x(s), \int_0^t k\left(s, x(\tau), x(v(\tau))\right) d\tau\right) \right|}{(t-s)^{1-\alpha}} ds$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t \frac{h(t)}{(t-s)^{1-\alpha}} ds \leq \frac{v(t)}{\Gamma(\xi)}$$

Taking supremum over t, we obtain

$$\|\mathbb{B}x\| \leq \frac{v(t)}{\Gamma(\xi)} = \mathcal{K}_1, \forall \, t \in \mathbb{R}_+$$

Therefore$\|\mathbb{B}x\| \leq \mathcal{K}_1, , \forall \, t \in \mathbb{R}_+$. Hence $\mathbb{B}(\mathbb{S})$ is a uniformly bounde subset of$\mathbb{X}$.

Now we will show that $\mathbb{B}(\mathbb{S})$ is equicontinuous set in $\mathbb{X}$. Let $t_1, t_2 \in \mathbb{R}_+$ with $t_1 > t_1$ and $x \in \mathbb{S}$, then we have

$$|\mathbb{B}x(t_1) - \mathbb{B}x(t_2)| = \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \frac{g\left(t_1, x(s), \int_0^t k\left(s, x(\tau), x(v(\tau))\right) d\tau\right) ds}{(t_1 - s)^{1-\alpha}} ds - \frac{1}{\Gamma(\alpha)} \int_0^{t_2} \frac{g\left(t_2, x(s), \int_0^t k\left(s, x(\tau), x(v(\tau))\right) d\tau\right) ds}{(t_2 - s)^{1-\alpha}} ds \right|$$

$$\leq \frac{1}{\Gamma(\alpha)} \left\{ \int_0^{t_1} \frac{\left|g\left(t_1, s, x(\gamma_1(t_1)), x(\gamma_2(t_1))\right)\right|}{(t_1 - s)^{1-\alpha}} ds - \frac{1}{\Gamma(\alpha)} \int_0^{t_2} \frac{\left|g\left(t_2, s, x(\gamma_1(t_2)), x(\gamma_2(t_2))\right)\right|}{(t_2 - s)^{1-\xi}} ds \right\}$$

$$\leq \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_1} \frac{h(t_1)}{(t_1 - s)^{1-\alpha}} ds - \int_0^{t_2} \frac{h(t_2)}{(t_2 - s)^{1-\alpha}} ds \right|$$

$$\leq \frac{1}{\Gamma(\alpha)} |v(t_1) - v(t_2)| \to 0 \text{ as } t_1 \to t_2$$

This shows that $\mathbb{B}(\mathbb{S})$ is an equicontinuous set in $\mathbb{X}$.

Hence $\mathbb{B}(\mathbb{S})$ is an equicontinuous set in $\mathbb{X} = \mathcal{AC}(\mathbb{R}_+, \mathbb{R})$ and so $\mathbb{B}(\mathbb{S})$ relatively compact by the Arzela-Ascoli Theorem. As a result, $\mathbb{B}$ is continuous and compact operator on $\mathbb{S}$.

Therefore by Dugundji and Granasthat $\mathbb{B}$ is completely continuous operator on $\mathbb{S}$.

Step III: The hypothesis (c) of theorem (3.1) is satisfies.

Let $x \in \mathbb{X}$ and $y \in \mathbb{S}$ be arbitrary elements such that $x = \mathbb{A}x\mathbb{B}y + \mathbb{C}x$ then we have

$|x(t)| = |\mathbb{A}x(t)\mathbb{B}x(t) + \mathbb{C}x(t)|$

$\leq |\mathbb{A}x(t)||\mathbb{B}x(t)| + |\mathbb{C}x(t)|$

$$\leq \left| f\left(t, x(t), x(u(t))\right) \right| \left| \frac{1}{\Gamma(\alpha)} \int_0^t \frac{g\left(t, x(s), \int_0^t k\left(s, x(\tau), x(v(\tau))\right) d\tau\right) ds}{(t - s)^{1-\alpha}} \right| +$$

$$\left| \sum_{i=1}^{i=n} I^{\beta_i} q_i\left(t, x(t)\right) ds \right|$$

$$\leq \left| f\left(t, x(t), x(u(t))\right) \right| \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\left|g\left(t, x(s), \int_0^t k\left(s, x(\tau), x(v(\tau))\right) d\tau\right)\right|}{(t - s)^{1-\alpha}} ds$$

$$+ \sum_{i=1}^n \int_0^t \frac{(t - s)^{\beta_i - 1}}{\Gamma(\beta_i)} |q_i(s, x(s))| ds$$

$$\leq \mathbb{F} \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} h(t) ds + \sum_{i=1}^n \int_0^t \frac{(t - s)^{\beta_i - 1} \|\lambda_i\|}{\Gamma(\beta_i)}$$

$$\leq \mathbb{F} \frac{v(t)}{\Gamma(\alpha)} + \|\lambda_i\| \sum_{i=1}^n \frac{\mathbb{T}^{\beta_i}}{\Gamma(\beta_i + 1)}$$

Which leads to

$\|x\| \leq \mathbb{F}\mathcal{K}_1 + \|\gamma\| \leq r$, where $\|\gamma\| = \|\lambda_i\| \sum_{i=1}^n \frac{\mathbb{T}^{\beta_i}}{\Gamma(\beta_i + 1)}$

Therefore $x \in \mathbb{S}$.

Step IV: Finally we show that $\zeta M + \eta < 1$ that is condition (d) of theorem (3.1) holds.

Since $M = \|\mathbb{B}(\mathbb{S})\| = sup_{x\in\mathbb{S}}\{sup_{t\in\mathbb{R}_+}|\mathbb{B}x(t)|\}$

$$= sup_{x\in\mathbb{S}}\left\{sup_{t\in\mathbb{R}_+}\left|\frac{1}{\Gamma(\alpha)}\int_0^t \frac{g\left(t,x(s),\int_0^t k\left(s,x(\tau),x(v(\tau))\right)d\tau\right)}{(t-s)^{1-\alpha}}ds\right|\right\}$$

$$\leq sup_{x\in\mathbb{S}}\left\{\frac{1}{\Gamma(\alpha)}\int_0^t \frac{\left|g\left(t,x(s),\int_0^t k\left(s,x(\tau),x(v(\tau))\right)d\tau\right)\right|}{(t-s)^{1-\alpha}}ds\right\}$$

$$\leq sup_{x\in\mathbb{S}}\left\{\frac{1}{\Gamma(\alpha)}\int_0^t \frac{h(t)}{(t-s)^{1-\alpha}}ds\right\}$$

$$\leq sup_{x\in\mathbb{S}}\left\{\frac{v(t)}{\Gamma(\alpha)}\right\} = \mathcal{K}_1$$

and therefore $\zeta M + \eta$, we have $(\mathbb{L}\mathcal{K}_1 + \|\gamma\|) < 1$, Where $\zeta = \mathbb{L}$ and $\eta = \|\gamma\| = \sum_{i=1}^n \|\lambda_i\| \frac{\mathbb{T}^{\beta_i}}{\Gamma(\beta_i+1)}$

Thus all the conditions of theorem (3.1) are satisfied and hence the operator equation $x = \mathbb{A}x\mathbb{B}y + \mathbb{C}x$ has a solution in $\mathbb{S}$.

Step IV: Finally we have to show that the locally attractivity of the solution for FNFIDE (1.1). Let x and y be two solutions of FNFIDE (1.1) in $\mathbb{S}$ defined on $\mathbb{R}_+$. Then we have

$|x(t) - y(t)|$

$$= \left|\left\{\begin{array}{c}\left(f\left(t,x(t),x(u(t))\right)\left[\frac{1}{\Gamma(\alpha)}\int_0^t \frac{g\left(t,x(s),\int_0^t k\left(s,x(\tau),x(v(\tau))\right)d\tau\right)ds}{(t-s)^{1-\alpha}}\right]\right) \\ +\sum_{i=1}^{i=n} I^{\beta_i}q_i\left(t,x(t)\right)\end{array}\right\} - \left\{\begin{array}{c}\left(f\left(t,y(t),y(u(t))\right)\left[\frac{1}{\Gamma(\alpha)}\int_0^t \frac{g\left(t,y(s),\int_0^t k\left(s,y(\tau),y(v(\tau))\right)d\tau\right)ds}{(t-s)^{1-\alpha}}\right]\right) \\ +\sum_{i=1}^{i=n} I^{\beta_i}q_i\left(t,y(t)\right)\end{array}\right\}\right|$$

$$\leq \left| f\left(t, x(t), x(u(t))\right) \left(\frac{1}{\Gamma(\alpha)} \int_0^t \frac{g\left(t, x(s), \int_0^t k\left(s, x(\tau), x(v(\tau))\right) d\tau\right) ds}{(t-s)^{1-\alpha}} \right) \right.$$

$$\left. + \sum_{i=1}^{i=n} I^{\beta_i} q_i\left(t, x(t)\right) \right|$$

$$+ \left| f\left(t, y(t), y(u(t))\right) \left(\frac{1}{\Gamma(\alpha)} \int_0^t \frac{g\left(t, y(s), \int_0^t k\left(s, y(\tau), y(v(\tau))\right) d\tau\right) ds}{(t-s)^{1-\alpha}} \right) \right.$$

$$\left. + \sum_{i=1}^{i=n} I^{\beta_i} q_i\left(t, y(t)\right) \right|$$

$$\leq \left| f\left(t, x(t), x(u(t))\right) \right| \left(\frac{1}{\Gamma(\alpha)} \int_0^t \frac{\left| g\left(t, x(s), \int_0^t k\left(s, x(\tau), x(v(\tau))\right) d\tau\right) \right| ds}{(t-s)^{1-\alpha}} \right)$$

$$+ \sum_{i=1}^{i=n} \frac{(t-s)^{\beta_i-1}}{\Gamma(\beta_i)} \left| q_i\left(s, x(s)\right) \right| ds$$

$$+ \left| f\left(t, y(t), y(u(t))\right) \right| \left(\frac{1}{\Gamma(\alpha)} \int_0^t \frac{\left| g\left(t, y(s), \int_0^t k\left(s, y(\tau), y(v(\tau))\right) d\tau\right) \right| ds}{(t-s)^{1-\alpha}} \right)$$

$$+ \sum_{i=1}^{i=n} \frac{(t-s)^{\beta_i-1}}{\Gamma(\beta_i)} \left| q_i\left(s, y(s)\right) \right| ds$$

$$\leq \mathbb{F} \left(\frac{1}{\Gamma(\alpha)} \int_0^t \frac{h(t) ds}{(t-s)^{1-\alpha}} \right)$$

$$+ \sum_{i=1}^{i=n} \frac{(t-s)^{\beta_i-1}}{\Gamma(\beta_i)} Q_i(t) ds + \mathbb{F} \left(\frac{1}{\Gamma(\alpha)} \int_0^t \frac{h(t) ds}{(t-s)^{1-\alpha}} \right) + \sum_{i=1}^{i=n} \frac{(t-s)^{\beta_i-1}}{\Gamma(\beta_i)} Q_i(t) ds$$

Taking supremum over t, we obtain

$$|x(t) - y(t)| \leq 2\mathbb{F} \frac{v(t)}{\Gamma(\alpha)} + 2\sum_{i=1}^{i=n} \frac{(t-s)^{\beta_i-1}}{\Gamma(\beta_i)} Q_i(t) ds$$

$$\leq 2\mathbb{F} \frac{v(t)}{\Gamma(\alpha)} + 2Q_i(t) \sum_{i=1}^{i=n} \frac{(t-s)^{\beta_i-1}}{\Gamma(\beta_i)} ds$$

$$\leq 2\mathbb{F} \frac{v(t)}{\Gamma(\alpha)} + 2Q_i(t) \sum_{i=1}^{n} \left[-\frac{(t-s)^{\beta_i}}{\Gamma(\beta_i)\beta_i} \right]_0^t$$

$$\leq 2\mathbb{F} \frac{v(t)}{\Gamma(\alpha)} + 2Q_i(t) \sum_{i=1}^{n} \left[\frac{(t)^{\beta_i}}{\Gamma(\beta_i+1)} \right]$$

$$\leq 2\mathbb{F} \frac{v(t)}{\Gamma(\alpha)} + 2Q_i(t) \sum_{i=1}^{n} \frac{\mathbb{T}^{\beta_i}}{\Gamma(\beta_i+1)}$$

$$\therefore |x(t) - y(t)| \leq 2\mathbb{F} \frac{v(t)}{\Gamma(\alpha)} + 2Q_i(t) \sum_{i=1}^{n} \frac{\mathbb{T}^{\beta_i}}{\Gamma(\beta_i+1)}, \forall\, t \in \mathbb{R}_+$$

Since $\lim_{t\to\infty} v(t) = 0, \lim_{t\to\infty} Q_i(t) = 0.$ Takng the limit superior as $t \to \infty$ in the above inequality yields, $\lim_{t\to\infty} |x(t) - y(t)| = 0$ .Therefore, there is a real number $T > 0$ such that $|x(t) - y(t)| < \epsilon$ for all $t \geq T$. Consequently, the solutions of FNFIDE (1.1) are locally attractive on $\mathbb{R}_+$.

This complete the proof.

5 Existence of extremal solutions:

We need following definitions in the sequel.

A closed and non-empty set $\mathbb{K}$ in a Banach Algebras $\mathbb{X}$ is called a Cone if

   i. $\mathbb{K} + \mathbb{K} \subseteq \mathbb{K}$

   ii. $\lambda \mathbb{K} \subseteq \mathbb{K}$ for $\lambda \in \mathbb{R}, \lambda \geq 0$

   iii. $\{-\mathbb{K}\} \cap \mathbb{K} = 0$ where 0 is the zero element of $\mathbb{X}$.

And a Cone is called positive Cone if

   iv. $\mathbb{K} \circ \mathbb{K} \subseteq \mathbb{K}$

and the notation $\circ$ is a multiplication composition in $\mathbb{X}$

We introduce an order relation $\leq$ in $\mathbb{X}$ as follows.

Let $x, y \in \mathbb{X}$ then $x \leq y$ if and only if $y - x \in \mathbb{K}$. A cone $\mathbb{K}$ is called normal if the norm $\|\cdot\|$ is monotone increasing on $\mathbb{K}$, there is a constant $N > 0$ such that $\|x\| \leq N\|y\|$ for all $x, y \in$ It is known that if the cone $\mathbb{K}$ is normal in $\mathbb{X}$ then every order-bounded set in $\mathbb{X}$ is norm-bounded set in $\mathbb{X}$. These concepts appear in the works of Heikkilaand Lakshmikantham [7].

We equip the space $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$ of continuous real valued function on $\mathbb{R}_+$ with the order relation $\leq$ with the help of cone defined by, $\mathbb{K} = \{x \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}): x(t) \geq 0 \ \forall t \in \mathbb{R}_+\}$

$$(5.1)$$

We well known that the cone $\mathbb{K}$ is normal and positive in $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$. As a result of positivity of the cone $\mathbb{K}$ we have:

Lemma 5.1(Dhage[2]): Let $\mathbb{K}$ be a positive cone in a real Banach Algebras $\mathbb{X}$ and let $u_1, u_2, v_1, v_2 \in \mathbb{K}$ be such that $u_1 \leq v_1, u_2 \leq v_2$ then $u_1 u_2 \leq v_1 v_2$.

Definition 5.1[2]: A mapping $\mathbb{Z}: [p, q] \to \mathbb{X}$ is said to be nondecreasing or monotone increasing if $x \leq y$ implies $\mathbb{Z}x \leq \mathbb{Z}y$ for all $x, y \in [p, q]$.

We use following fixed point theorem of Dhage [9] for proving the existence of extremal solution for the FNFIDE (1.1) under certain monotonicity conditions.

Theorem 5.1 [9]: Let $\mathbb{K}$ be a cone in Banach Algebra $\mathbb{X}$ and let $[p, q] \in \mathbb{X}$. Suppose that $\mathbb{A}, \mathbb{B}: [p, q] \to \mathbb{K}$ and $\mathbb{C}: [p, q] \to \mathbb{X}$ be three nondecreasing operators such that

   a. $\mathbb{A}$ and $\mathbb{C}$ are a Lipschitz with Lipschitz constant α, β

   b. $\mathbb{B}$ is completely continuous,

   c.   The elements $p, q \in \mathbb{X}$ satisfy $p \leq \mathbb{A}p\mathbb{B}p + \mathbb{C}p$ and $\mathbb{A}q\mathbb{B}q + \mathbb{C}q \leq q$

Further if the cone $\mathbb{K}$ is normal and positive then the operator equation $x = \mathbb{A}x\mathbb{B}y + \mathbb{C}x$ has the least and greatest positive solution in $[p, q]$ whenever $\alpha M + \beta < 1$, where $M = \|\mathbb{B}([p, q])\| = sup\{\|\mathbb{B}x\|: x \in [p, q]\}$.

Definition 5.2: A function $p \in \mathcal{BC}(\mathbb{R}_+, \mathbb{R})$ is called a lower solution of the FNFIDE (1.1) on $\mathbb{R}_+$ if the function $t \to \left\{\dfrac{p(t) - \sum_{i=1}^{i=n} I^{\beta_i} q_i(t, p(t))}{f(t, p(t), x(p(t)))}\right\}$ is continuous and

$$\left. \begin{array}{c} \dfrac{d^\xi}{dt^\xi}\left\{\dfrac{p(t) - \sum_{i=1}^{i=n} I^{\beta_i} q_i(t, p(t))}{f(t, p(t), x(p(t)))}\right\} \leq g\left(t, p(t), \int_0^t k\left(t, p(s), p(v(s))\right) ds\right) \\ p(0) \leq 0 \end{array} \right\} (5.1)$$

Again a function $q \in \mathcal{BC}(\mathbb{R}_+, \mathbb{R})$ is called an upper solution of the FNFIDE(3.1.1) on $\mathbb{R}_+$ if the function $t \to \left\{ \dfrac{q(t) - \sum_{i=1}^{n} I^{\beta_i} q_i(t, q(t))}{f\left(t, q(\mu_1(t)), q(\mu_2(t))\right)} \right\}$ is continuous and

$$\left. \begin{array}{c} \dfrac{d^{\xi}}{dt^{\xi}} \left\{ \dfrac{q(t) - \sum_{i=1}^{i=n} I^{\beta_i} q_i(t, q(t))}{f\left(t, q(t), x(q(t))\right)} \right\} \geq g\left(t, q(t), \int_0^t k\left(t, q(s), q(v(s))\right) ds\right) \\ q(0) \geq 0 \end{array} \right\} (5.2)$$

Definition 5.3: A solution x_M of the FNFIDE (1.1) is said to be maximal if for any other solution x to FNFIDE (1.1) one has $x(t) \leq x_M(t)$ for all $t \in \mathbb{R}_+$. Again a solution x_M of the FNFIDE (1.1) is said to be minimal if $x_M(t) \leq x(t)$ for all $t \in \mathbb{R}_+$ where x is any solution of the FNFIDE (1.1) on $\mathbb{R}_+$.

Remark 5.1: The maximal and minimal solutions of the FNFIDE (1.1) are respectively the upper and lower solutions of FNFIDE (1.1) and ice-versa.

Definition 5.4: (Caratheodory case) A function $d: \mathbb{R} \to \mathbb{R}$ is nondecreasing if $d(x) \leq d(y) \ \forall x, y \in \mathbb{R}$ for which $x \leq y$. Similarly, $d(x)$ is increasing in x if $d(x) < d(y) \ \forall x, y \in \mathbb{R}$ for which $x < y$.

We consider the following assumptions:

$(\mathcal{H}_6)$ The function $x \to \left\{ \dfrac{x(t) - \sum_{i=1}^{n} I^{\beta_i} q_i(t, x(t))}{f\left(t, x(t), x(u(t))\right)} \right\}$ is increasing in the interval $\left[\min_{t \in \mathbb{R}_+} p(t), \max_{t \in \mathbb{R}_+} q(t) \right]$.

$(\mathcal{H}_7)$ The functions $f: \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} - \{0\}, g: \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $q_i: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ are nondecreasing in x almost everywhere for $t \in \mathbb{R}_+$

$(\mathcal{H}_8)$ The FNFIDE (1.1) has a lower solution p and upper solution q on $\mathbb{R}_+$ with $p \leq q$.

$(\mathcal{H}_9)$ The function g is caratheodory.

$(\mathcal{H}_{10})$ The function $l: \mathbb{R}_+ \to \mathbb{R}$ defined by
$l(t) = \left| g\left(t, p(t), \int_0^t k\left(t, p(s), p(v(s))\right) ds\right) \right| + \left| g\left(t, q(t), \int_0^t k\left(t, q(s), q(v(s))\right) ds\right) \right|$ is Lebesgue measurable.

Remark 5.2: Assume that the hypotheses $(\mathcal{B}_1\text{-}\mathcal{B}_4)$ holds, then the function $t \to g\left(t, x(t), \int_0^t k\left(t, x(s), x(v(s))\right) ds\right)$ is lebesgue integrable on $\mathbb{R}_+$, say $\left| g\left(t, x(t), \int_0^t k\left(t, x(s), x(v(s))\right) ds\right) \right| \leq l(t)$, a.e., $t \in \mathbb{R}_+$ for all $x \in [p, q]$ and some lebesgueintegrable function l.

Theorem 5.2: Assume that the hypothesis $(\mathcal{H}_1 - \mathcal{H}_{10})$ holds and l is given in above remark(5.2), further $\mathbb{L}\left[\dfrac{\mathbb{T}^{\alpha}}{\Gamma(\alpha+1)} \right] \|l\| + \|\gamma\| \leq 1$, then FNFIDE (1.1) has minimal and maximal positive solution on $\mathbb{R}_+$.

Proof: The FNFIDE (1.1) is equivalent to IE (3.1) on $\mathbb{R}_+$. Let $\mathbb{X} = \mathcal{C}(\mathbb{R}_+, \mathbb{R})$ and we define an order relation "$\leq$" by the cone $\mathbb{K}$ given by (5.1). Clearly $\mathbb{K}$ is a normal cone in $\mathbb{X}$. Define three operators $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ on $\mathbb{X}$ by (4.2), (4.3) and (4.4) respectively. Then IE (3.1) is transformed into an operator equation $\mathbb{A}x\mathbb{B}x + \mathbb{C}x = x$ in Banachalgebra$\mathbb{X}$. Notice that $(\mathcal{H}_7)$ implies $\mathbb{A}, \mathbb{B}: [p, q] \to \mathbb{K}$ also note that $(\mathcal{H}_8)$ ensures that $p \leq \mathbb{A}p\mathbb{B}p + \mathbb{C}p$ and $\mathbb{A}q\mathbb{B}q + \mathbb{C}q \leq q$. Since the cone $\mathbb{K}$ in $\mathbb{X}$ is

normal, $[p, q]$ is a norm bounded set in $\mathbb{X}$. Now it is shown, as in the proof of Theorem (3.4.1), that $\mathbb{A}$ and $\mathbb{C}$ are Lipschitz with a Lipschitz constant $\|\alpha\|$ and $\|\beta\|$ respectively. Similarly $\mathbb{B}$ is completely continuous operator on $[p, q]$. Again the hypothesis $(\mathcal{H}_8)$ implies that $\mathbb{A}, \mathbb{B}$ and $\mathbb{C}$ are non-decreasing on $[p, q]$. To see this, let $x, y \in [p, q]$ be such that $x \leq y$. Then by $(\mathcal{H}_7)$,

$$\mathbb{A}x(t) = f\left(t, x(t), x(u(t))\right) \leq f\left(t, y(t), y(u(t))\right) \leq \mathbb{A}y(t), \forall\, t \in \mathbb{R}_+$$

and, $\mathbb{B}x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{g\left(t, x(s), \int_0^t k\left(t, x(\tau), x(v(\tau))\right)d\tau\right)ds}{(t-s)^{1-\alpha}}$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t \frac{g\left(t, y(s), \int_0^t k\left(t, y(\tau), y(v(\tau))\right)d\tau\right)ds}{(t - s)^{1-\alpha}} ds \leq \mathbb{B}y(t), \forall t \in \mathbb{R}_+$$

And $\mathbb{C}x(t) = \sum_{i=1}^{i=n} I^{\beta_i} q_i\left(t, x(t)\right) = \sum_{i=1}^{n} \int_0^t \frac{(t-s)^{\beta_i-1}}{\Gamma(\beta_i)} q_i\left(s, x(s)\right)ds$

$$\leq \sum_{i=1}^{n} \int_0^t \frac{(t-s)^{\beta_i-1}}{\Gamma(\beta_i)} q_i\left(s, y(s)\right)ds \leq \mathbb{C}y(t), \forall\, t \in \mathbb{R}_+$$

Implies that $\mathbb{A}, \mathbb{B}$ and $\mathbb{C}$ are nondecreasing operators on $[p, q]$.

Again definition (5.2) and hypothesis $(\mathcal{H}_8)$ implies that

$$p(t) \leq f\left(t, p(t), p(u(t))\right)\left[\frac{1}{\Gamma(\alpha)} \int_0^t \frac{g\left(t, p(s), \int_0^t k\left(s, p(\tau), p(v(\tau))\right)d\tau\right)ds}{(t-s)^{1-\alpha}}\right]$$

$$+ \sum_{i=1}^{i=n} I^{\beta_i} q_i\left(t, p(t)\right)$$

$$\leq f\left(t, x(t), x(u(t))\right)\left[\frac{1}{\Gamma(\alpha)} \int_0^t \frac{g\left(t, x(s), \int_0^t k\left(s, x(\tau), x(v(\tau))\right)d\tau\right)ds}{(t - s)^{1-\alpha}}\right]$$

$+ \sum_{i=1}^{i=n} I^{\beta_i} q_i\left(t, x(t)\right)$

$$\leq f\left(t, q(t), q(u(t))\right)\left[\frac{1}{\Gamma(\alpha)} \int_0^t \frac{g\left(t, q(s), \int_0^t k\left(s, q(\tau), q(v(\tau))\right)d\tau\right)ds}{(t - s)^{1-\alpha}}\right]$$

$+ \sum_{i=1}^{i=n} I^{\beta_i} q_i\left(t, q(t)\right)$
$\leq q(t), \forall\, t \in \mathbb{R}_+$ and $x \in [p, q]$

As a result $p(t) \leq \mathbb{A}x(t)\mathbb{B}x(t) + \mathbb{C}x(t) \leq q(t), \forall t \in \mathbb{R}_+$ and $x \in [p, q]$

Hence $\mathbb{A}x\mathbb{B}x + \mathbb{C}x \in [p, q], \forall x \in [p, q]$

Again $M = \|\mathbb{B}([p, q])\| = sup\{\|\mathbb{B}x\|: x \in [p, q]\}$

$$\leq sup\left\{ sup_{t \in \mathbb{R}_+} \frac{1}{\Gamma(\alpha)} \int_0^t \frac{g\left(t, x(s), \int_0^t k\left(t, x(\tau), x(v(\tau))\right)d\tau\right)ds}{(t - s)^{1-\alpha}}: x \in [p, q]\right\}$$

$$\leq sup_{x \in [p, q]} \left\{ \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\left|g\left(t, x(s), \int_0^t k\left(s, x(\tau), x(v(\tau))\right)d\tau\right)\right|}{(t - s)^{1-\alpha}} ds\right\}$$

$$\leq sup_{x\in[p,q]}\left\{\frac{1}{\Gamma(\alpha)}\int_0^t \frac{l(t)}{(t-s)^{1-\alpha}}ds\right\}$$

$$\leq sup_{x\in[p,q]}\left\{\frac{1}{\Gamma(\alpha)}\int_0^t \frac{\|l\|}{(t-s)^{1-\alpha}}ds\right\}$$

$$\leq sup_{x\in[p,q]}\left\{\frac{1}{\Gamma(\alpha)}\int_0^t \frac{\|l\|}{(t-s)^{1-\alpha}}ds\right\}$$

$$\leq sup_{x\in[p,q]}\left\{\left[-\frac{(t-s)^\alpha}{\Gamma(\alpha)\alpha}\right]_0^t \|l\|\right\}$$

$$\leq sup_{x\in[p,q]}\left\{\left[\frac{(t)^\alpha}{\Gamma(\alpha+1)}\right]\|l\|\right\}$$

$$\leq \left\{\left[\frac{\mathbb{T}^\alpha}{\Gamma(\alpha+1)}\right]\|l\|\right\}$$

Since $\alpha M + \beta \leq \mathbb{L}\left[\frac{\mathbb{T}^\alpha}{\Gamma(\alpha+1)}\right]\|l\| + \|\gamma\| \leq 1$

We apply theorem (5.1) to the operator equation $\mathbb{A}x\mathbb{B}x + \mathbb{C}x = x$ to yield that the FNFIDE (1.1) has minimum and maximum positive solution on $\mathbb{R}_+$.

This completes the proof

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